A Critique on Average-Case Noise Analysis in RLWE-Based Homomorphic Encryption

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WAHC' 25

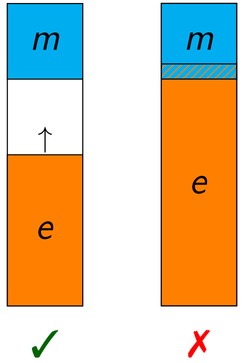
Recap: RLWE-based Homomorphic Encryption

Secret key s Message Δm Noise e

$$(as + \Delta m + e, a)$$

- Polynomial ring $\mathcal{R} = \mathbb{Z}[X]/(X^N + 1)$
- a, s, m, e are polynomials
- Poly multiplication is coeff convolution

$$a \cdot b = \sum_{i=0}^{2N-1} \left(\sum_{j+k=i} a_j b_k \right) X^i \pmod{X^N + 1}$$

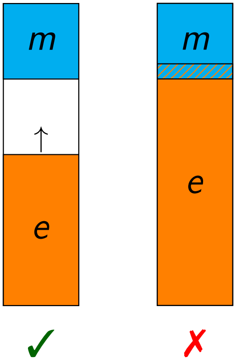


Recap: RLWE-based Homomorphic Encryption

Secret key s Message Δm Noise e

$$(as + \Delta m + e, a)$$

- Message in high bits
- Noise grows after HE operations
- Noise should never *overflow*

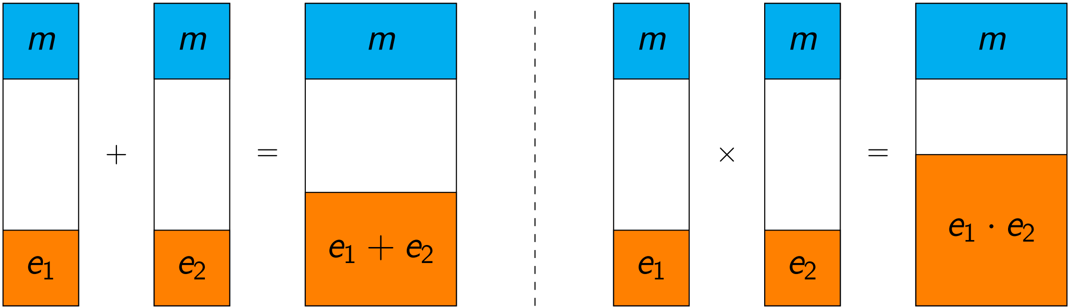


$$\text{correctness} \xleftrightarrow{\textit{tradeoff}} \text{efficiency}$$

$$\Downarrow$$

Noise Analysis

Worst-Case Noise Analysis



$+$ **is easy**

$$\|e_1 + e_2\| \leq \|e_1\| + \|e_2\|$$

$\times$ **is hard**

$$\|e_1 \cdot e_2\| \leq N \cdot \|e_1\| \cdot \|e_2\|$$

- N is *worst-case expansion factor*, very conservative and undesired
- Empirically, growth should be $C\sqrt{N}$

Average-Case Noise Analysis: Variance-Based

$$e \longrightarrow \mathrm{Var}(e) \xrightarrow[\text{Gaussian Heuristic (✗)}]{} \|e\| = 6\sigma$$

Previous Workflow

$$e \longrightarrow \mathrm{Var}(e) \xrightarrow[\text{Heavier tail}]{} \text{Larger } \|e\|$$

Our Observation

- [CCH+24]: Noises follow *Gaussian* distribution
  - Estimate the variance $\mathrm{Var}(e)$ after each step
  - Finally induce the *Gaussian* bound $\|e\|$
- Contribution 1: Invalidate the Gaussian Heuristic
  - Noises are not Gaussian after deep multiplications

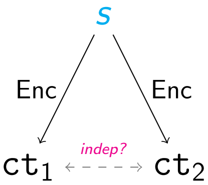
$$\mathrm{Var}(e_1 + e_2) = \mathrm{Var}(e_1) + \mathrm{Var}(e_2)$$

$$\mathrm{Var}(e_1 \cdot e_2) = N \cdot \mathrm{Var}(e_1) \cdot \mathrm{Var}(e_2)$$

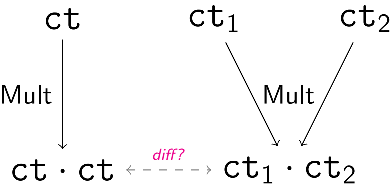
$$\Downarrow ✗$$

$$\|e_1 \cdot e_2\| = \sqrt{N} \cdot \|e_1\| \cdot \|e_2\|$$

Dependencies



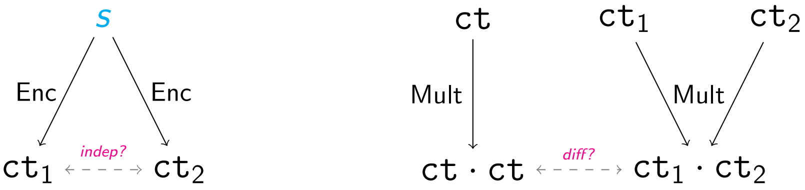
Common Dependencies



Ciphertext Dependencies

- Contribution 2: Study *dependencies* in noise analysis
- Previously, use Independence Heuristic because dependencies are hard
- Two types of dependencies
  - Common dependencies: All $\mathtt{ct}$ share secret key s
  - Ciphertext dependencies: $\mathtt{ct} \cdot \mathtt{ct}$ *v.s.* $\mathtt{ct}_1 \cdot \mathtt{ct}_2$

Dependencies



Common Dependencies   Ciphertext Dependencies

- Contribution 2: Study *dependencies* in noise analysis
- Previously, use Independence Heuristic because dependencies are hard
- Two types of dependencies
  - Common dependencies: All $\mathtt{ct}$ share secret key s
  - Ciphertext dependencies: $\mathtt{ct} \cdot \mathtt{ct}$ *v.s.* $\mathtt{ct}_1 \cdot \mathtt{ct}_2$
- Contribution 3: Find flaws in OpenFHE empirical formula
  - Root cause: Gaussian Heuristic + Independence Heuristic
  - Special cases will violate both

Section 1

Technical Details

Noise in BFV

- Polynomial ring $\mathcal{R} = \mathbb{Z}[X]/(X^N + 1)$, N power of 2
- Noise $e \leftarrow \mathcal{N}(0, \sigma^2)$ with Gaussian coefficients
- Secret key $s \leftarrow \{-1, 0, 1\}$ with uniform ternary coefficients

$$
\begin{aligned}
\mathtt{pk} &= (-as + e_{\mathtt{pk}}, a) \\
\mathtt{ct} &= u_{\mathtt{ct}} \cdot \mathtt{pk} + (\Delta m + e_{\mathtt{ct}}, e'_{\mathtt{ct}}) \\
\mathtt{ct}(s) &= \Delta m + \underbrace{e_{\mathtt{ct}} + u_{\mathtt{ct}} \cdot e_{\mathtt{pk}} + e'_{\mathtt{ct}} \cdot s}_{\text{Noise } v}
\end{aligned}
$$

- Common dependencies: secret key s and noise in public key $e_{\mathtt{pk}}$
- Ciphertext dependencies: ciphertext specific $u_{\mathtt{ct}}$ and $e_{\mathtt{ct}}$

BFV Multiplication

BFV $\mathtt{ct}$ in $\mathcal{R}/Q\mathcal{R}$. Multiplication happens in $\mathcal{R}$.

$$
\begin{aligned}
\mathtt{ct}_1(s) &= \Delta m_1 + v_1 + h_1 Q \\
h_1 Q &\approx c_1 \boxed{s} \\
h_1 &\approx \mu_{\mathtt{ct}_1} \boxed{s} \\
(\mathtt{ct}_1 \otimes \mathtt{ct}_2)(s) &= v_1 h_2 + v_2 h_1 + \cdots \\
&= \boxed{s^2} \cdot (\mu_{\mathtt{ct}_2} e_{\mathtt{ct}_1}) + \cdots
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$$

Assume relinearize after each multiplication
but introduces negligible noise

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k-way multiplication contains high degree terms

- s^k in noise *generally*
- $\mu_{\texttt{ct}}^{k-1}$ in noise in specific circuit

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Need to study distribution of

- Product of Gaussians: $\prod f_i$
- Power of one Gaussian: f^k
- Mixed Product of Gaussians: $\prod f_i^{k_i}$

Are they Gaussian?

$\Rightarrow$ Study the Kurtosis!

Kurtosis/Bound of Gaussian

Definition (Kurtosis)

The **Kurtosis** of a zero-mean random variable X is defined as

$$\mathrm{Kurt}(X) = \frac{\mathbb{E}[X^4]}{(\mathbb{E}[X^2])^2} = \frac{\mathbb{E}[X^4]}{\mathrm{Var}(X)^2}$$

- Kurtosis measures *tailedness* [Wes14]
- Gaussian has *constant* Kurt $= 3$

$$\mathrm{Kurt}(\mathcal{N}(0,\sigma^2)) = \frac{3\sigma^4}{(\sigma^2)^2} = 3$$

- Usually bound X using $B = 6\sigma$

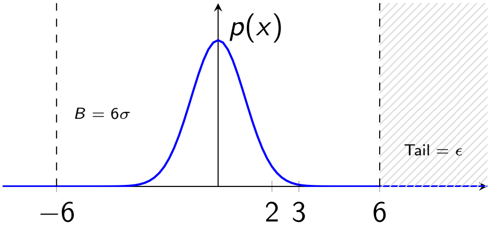
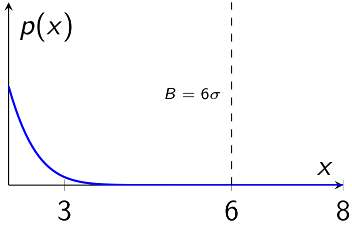


Figure: Gaussian Distribution with 6σ Bound

$$P(|X| > B) = \mathtt{erfc}\left(\frac{B}{\sigma\sqrt{2}}\right) \approx 2^{-28} = \epsilon$$

Kurtosis and Bound

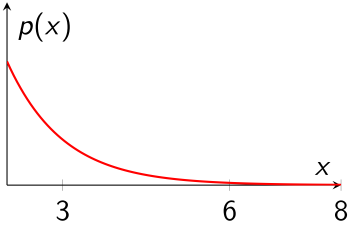


Gaussian Distribution

$p(x) \sim \exp(-x^2)$

$\text{Kurt} = 3$

$B = 6\sigma$

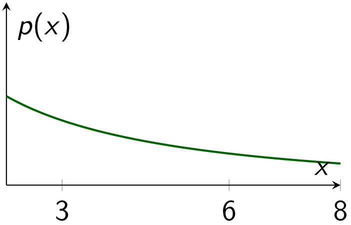


Laplace Distribution

$p(x) \sim \exp(-|x|)$

$\text{Kurt} = 6$

$B = 23.3\sigma$



Generalized Gaussian Distribution

$p(x) \sim \exp(-|x|^{0.5})$

$\text{Kurt} = 25.2$

$B = 43.5\sigma$

- Trend: Tail decays slower $\searrow$, Tail heavier $\nearrow$, Kurtosis $\nearrow$, Bound $\nearrow$
- Kurtosis $\ggg 3 \quad \Rightarrow \quad$ not Gaussian

B derived with the same failing probability ϵ

Main Theorems

Let $f_i \leftarrow \mathcal{N}(0,1)$ be independent Gaussian polynomials

Theorem ($\prod$ Indep)

$F = \prod_i^k f_i$

$$\mathrm{Kurt}(F) = 3 + 3\frac{2^k - 2}{N}$$

Theorem ($\prod$ Same)

$F = f^k$

$$\mathrm{Kurt}(F) = 3 + 3\frac{\binom{2k}{k} - 2}{N}$$

Theorem (Mixed $\prod$)

$F = \prod f_i^{k_i}$

$$\mathrm{Kurt}(F) = 3 + 3\frac{\prod \binom{2k_i}{k_i} - 2}{N}$$

- k is multiplication depth; N is ring dimension
- For practical $N = 2^{16}$, $k > 16$ (or $k > 10$) $\Rightarrow$ Kurt > 6 $\Rightarrow$ F not Gaussian

$\boxed{\textit{Noises are not Gaussian!}}$

in deep multiplications

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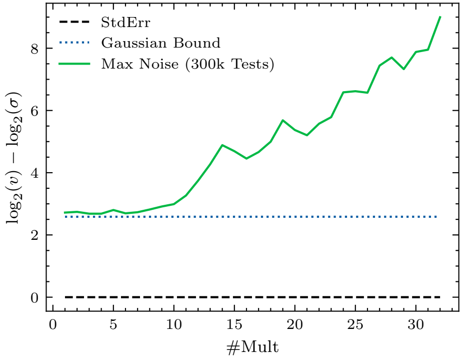
$$\mathrm{Kurt}(F) = 3 + 3\frac{\prod \binom{2k_i}{k_i} - 2}{N}$$

- Remark 1: When $N \to \infty$, Kurt $\to 3$
  - It becomes Gaussian!
  - Exactly Central Limit Theorem
  - But here N is *finite*, so CLT fails
- Remark 2: When k small, Kurt ≈ 3
  - Illusion that noises are *always* Gaussian
  - Past experiments only done for small k
- Remark 3: There is no widely-known name for these distributions

Experimental Results

Gaussian bound fails $\Rightarrow$ Noises not Gaussian

- x-axis: mul-depth k
- Calculate the variance σ^2 thus StdErr σ
- Sample 300k times and record max noise
- Normalize to 0 to see the difference
- y-axis: max noise v.s. StdErr
  - $\log_2(v/\sigma) = \log_2(v) - \log_2(\sigma)$
  - *In logarithm scale!*
- Gaussian bound: 6σ

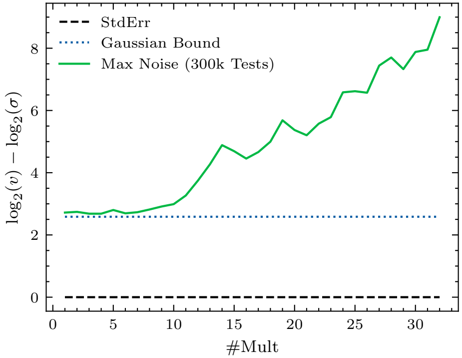


$$e_1 \to e_1 e_2 \longrightarrow \cdots \longrightarrow \prod^{32} e_i$$

Experimental Results

Gaussian bound fails $\Rightarrow$ Noises not Gaussian

- 8 bit overflow $\Rightarrow$ 256σ deviation
- If Gaussian, happens with prob 2^{-47282}
- When k small, noises are Gaussian-like
  - $\text{Kurt} = 3 + 3\frac{2^k - 2}{N}$
  - k small $\Rightarrow$ $\text{Kurt} \approx 3$
- Caused *illusion* that noises are Gaussian!



$$e_1 \to e_1 e_2 \longrightarrow \cdots \longrightarrow \prod^{32} e_i$$

Curve not smooth because of independent samples

Case study: How dependencies affact the variance

We also calculate the variance

Theorem ($\prod$ Indep)

$$Var\left(\prod f_i\right) = N^{k-1}$$

Theorem ($\prod$ Same)

$$Var\left(f^k\right) = k!N^{k-1}$$

Theorem (Mixed $\prod$)

$$Var\left(\prod f_i^{k_i}\right) = \prod k_i!N^{k-1}$$

BFV $\mathtt{ct}$ independent product

$$\mathrm{Var}\left(\left(\bigotimes_i^k \mathtt{ct}_i\right)(s) = \boxed{s^k} \cdot \left(\prod \mu_{\mathtt{ct}_i} e_{\mathtt{ct}_j}\right) + \cdots\right) \approx k! \cdot N^{2k-1} \cdot \mathrm{Var}(s)^k \cdots$$

We are able to exactly derive $k!$ while [BMCM23] used experimental correction factor

$$\boxed{\textit{Common dependencies} \quad \Rightarrow \quad \textit{Variance} \nearrow}$$

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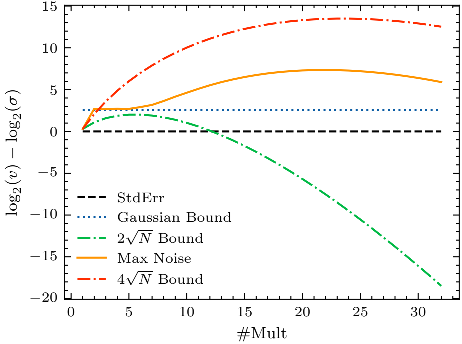
BFV ct dependent product *v.s.* independent product

$$\frac{\mathrm{Var}\left(\mathtt{ct}^k(s) = \boxed{s^k \mu_{\mathtt{ct}}^{k-1}} \cdot e_{\mathtt{ct}} + \cdots\right)}{\mathrm{Var}\left(\left(\bigotimes_i^k \mathtt{ct}_i\right)(s) = \boxed{s^k} \cdot \left(\prod \mu_{\mathtt{ct}_i} e_{\mathtt{ct}_j}\right) + \cdots\right)} \approx (k-1)!$$

$$\boxed{\textit{Ciphertext dependencies} \quad \Rightarrow \quad \textit{Variance} \nearrow}$$

Test the Empirical Formula in OpenFHE

- OpenFHE used $2\sqrt{N}$ *empirical* expansion factor
  - Originally tested using $e \cdot e'$ and $e \cdot s$
- Works for e^k, $\prod e_i$, $\prod s_i$
- Fails for s^k and modulus switching error
  - Does not affect security because of other loose factors
- Contacted OpenFHE and fixed in v1.3.1
  - Use $4\sqrt{N}$ for these special cases



s^k

Implications and Open Questions

- Software needs to track common dependencies and ciphertext dependencies
  - If they want to do average-case noise analysis
  - Means an *application-specific* analysis and parameter generation
  - Agrees with the Application-Aware security model [AAMP24]
  - Compiler can help here!

- What is the true distribution of the noises?
  - Noise analysis needs the bound!
  - We only calculated the kurtosis

$$\text{Kurt} \xrightarrow{\quad ? \quad} \text{Bound } B$$

- How can ciphertext dependencies be used for attack?
  - Recent attacks[1] [GNSJ24, CCP$^+$24, CSBB24] exploited such dependencies in addition $(+)$
  - We are able to analyse dependencies in multiplication $(\times)$.

[1] or misconfiguration as argued in [AAMP24]

Informal Comments on CKKS Average-Case Noise Analysis

- The major term[2] in CKKS noise is

$$m_1 \cdot m_2 \cdot m_3 \cdots (e_{\mathtt{ct}})$$

- We know little about messages m_i (otherwise security implications)
- Previous works assume m_i are *uniform in range* $[-1, 1]$
  - Assumption is not practical
- Need distribution analysis and range analysis depending on applications
- No good ways to do average-case
- Maybe we can only use worst-case analysis, or empirical results

[2] Especially for OpenFHE "reduced error" implementation

Summary

- Noises are not Gaussian after deep multiplications
- Dependencies greatly affect the variance and kurtosis of the noise
- Find flaws in empirical formula in OpenFHE

Thank you!

Questions?

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